

EXACTNESS OF REDUCED AMALGAMATED FREE PRODUCT C*-ALGEBRAS

KENNETH J. DYKEMA

1 November, 1999

ABSTRACT. Some completely positive maps on reduced amalgamated free products of C*-algebras are constructed, showing that every reduced amalgamated free product of exact C*-algebras is exact. Consequently, every amalgamated free product of exact discrete groups is exact.

INTRODUCTION.

A C*-algebra A is said to be exact if for every short exact sequence

$$0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$$

of C*-algebras and *-homomorphisms, the sequence

$$0 \rightarrow A \otimes J \rightarrow A \otimes B \rightarrow A \otimes (B/J) \rightarrow 0 \tag{1}$$

of spatial tensor products is exact. The issue was first raised when S. Wassermann [10] proved in an example that the sequence (1) need not be exact. Later, he showed [11] that a sufficient condition for A to be exact is that it have a nuclear embedding $A \hookrightarrow A'$ for some C*-algebra A' , i.e. that for arbitrary $\epsilon > 0$ and for every finite subset $\omega \subseteq A$ there be an integer n and completely positive contractions $\Phi : A \rightarrow M_n(\mathbb{C})$ and $\Psi : M_n(\mathbb{C}) \rightarrow A'$ such that $\|x - \Psi \circ \Phi(x)\| < \epsilon$ for every $x \in \omega$. The class of exact C*-algebras is known to be closed under taking subalgebras, taking spatial tensor products and taking inductive limits. In some remarkable papers, [3] and [4], E. Kirchberg proved a number of conditions equivalent to exactness for separable, unital C*-algebras, among them nuclear embeddability. However, in this article we will not need to make use of Kirchberg's powerful results. A good reference for exact C*-algebras is Wassermann's monograph [12].

1991 *Mathematics Subject Classification.* 46L05, 46L35.

Partially supported as an invited researcher funded by CNRS of France.

In [8], Voiculescu introduced the noncommutative probabilistic theory of freeness, which has turned out to be instrumental to the study of C^* -algebras and von Neumann algebras associated to free products of groups, (see for example the book [9]). His amalgamated or “ B -valued” version of freeness is as follows: if A is a unital C^* -algebra with a unital C^* -subalgebra B and a conditional expectation (i.e. a projection of norm 1), $\phi : A \rightarrow B$ and if $B \subseteq A_\iota \subseteq A$ are intermediate C^* -subalgebras, ($\iota \in I$), then the family $(A_\iota)_{\iota \in I}$ is said to be free with respect to ϕ if $\phi(a_1 a_2 \cdots a_n) = 0$ whenever $a_j \in A_{\iota_j} \cap \ker \phi$ and $\iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n$. Voiculescu also introduced the reduced amalgamated free product of C^* -algebras, which we now describe. If B is a unital C^* -algebra and if for a set I and every $\iota \in I$, A_ι is a unital C^* -algebra containing a copy of B as a unital C^* -subalgebra and having a conditional expectation $\phi_\iota : A_\iota \rightarrow B$ whose GNS representation is faithful (see 1.1 below), then there is a unique unital C^* -algebra A containing a unital copy of B , with a conditional expectation $\phi : A \rightarrow B$ and with embeddings $A_\iota \hookrightarrow A$ restricting to the identity on B such that

- (i) $\forall \iota \in I \phi|_{A_\iota} = \phi_\iota$;
- (ii) the family $(A_\iota)_{\iota \in I}$ is free with respect to ϕ ;
- (iii) A is generated by $\bigcup_{\iota \in I} A_\iota$;
- (iv) the GNS representation of ϕ is faithful on A .

This is called the reduced amalgamated free product of C^* -algebras, and is denoted by

$$(A, \phi) = \bigstar_{\iota \in I} (A_\iota, \phi_\iota). \quad (2)$$

In the case when $B = \mathbf{C}$, the conditional expectations are just states and the construction (2) is often called simply the reduced free product.

The following prominent example relates the reduced amalgamated free product of C^* -algebras to the amalgamated free product of groups. Let B be the reduced C^* -algebra $C_{\text{red}}^*(H)$ of a discrete group H , let G_ι be a discrete group containing a copy of H as a subgroup and Take the conditional expectation $\tau_H^{G_\iota} : A_\iota \rightarrow B$ given by

$$\tau_H^{G_\iota}(\lambda_g) = \begin{cases} \lambda_g & \text{if } g \in H \\ 0 & \text{if } g \notin H. \end{cases}$$

Then the reduced amalgamated free product construction yields

$$(C_{\text{red}}^*(G), \tau_H^G) = \bigstar_{\iota \in I} (C_{\text{red}}^*(G_\iota), \tau_H^{G_\iota}) \quad (3)$$

where G is the amalgamated free product of groups $G = (*_H)_{\iota \in I} G_\iota$.

In this paper it is proved that in the reduced amalgamated free product of C^* -algebras (2), if each A_ι is exact then A is exact. The proof proceeds by the construction of completely positive maps showing that A satisfies a condition analogous to nuclear embeddability; (see Lemma 2.1). Using different techniques, Kirchberg has proved [4] that every reduced amalgamated free product of finite dimensional C^* -algebras is exact.

In [5], Kirchberg and Wassermann defined a locally compact group G to be exact if the sequence

$$0 \rightarrow J \rtimes_{\alpha|_{J,r}} G \rightarrow A \rtimes_{\alpha,r} G \rightarrow (A/J) \rtimes_{\tilde{\alpha},r} G \rightarrow 0$$

is exact for every continuous action α of G on a C^* -algebra A and every α -invariant ideal J of A . They also showed that a discrete group G is exact if and only if its reduced group C^* -algebra $C_{\text{red}}^*(G)$ is exact. A consequence of our main result is that the class of exact discrete groups is closed under taking amalgamated free products.

In §1, Voiculescu's construction of the reduced amalgamated free product of C^* -algebras is described in detail. In §2, some preliminary lemmas about exact C^* -algebras and Hilbert C^* -modules are proved. In §3, completely positive maps on reduced amalgamated free product C^* -algebras are constructed and the main result is proved.

Acknowledgements. I would like to thank Etienne Blanchard and Marius Junge for several helpful conversations. Most of this work was done while I was visiting *l'Institut de Mathématiques de Luminy* near Marseille. I would like to thank the members of the department and especially Étienne Blanchard and Jérôme Chabert for their kind hospitality.

§1. THE CONSTRUCTION OF REDUCED AMALGAMATED FREE PRODUCTS.

In this section, by way of introducing some notation and a few conventions, we recall Voiculescu's construction [8] of reduced amalgamated free products of C^* -algebras.

Let B be a unital C^* -algebra, let I be a set having at least two elements and for every $\iota \in I$ let A_ι be a unital C^* -algebra containing a copy of B as a unital C^* -subalgebra; suppose that $\phi_\iota : A_\iota \rightarrow B$ is a conditional expectation satisfying the property that

$$\forall a \in A_\iota \setminus \{0\} \quad \exists x \in A_\iota \quad \phi_\iota(x^* a^* a x) \neq 0. \quad (4)$$

Then

$$(A, \phi) = \ast_{\iota \in I} (A_\iota, \phi_\iota)$$

denotes the reduced amalgamated free product of C^* -algebras, whose construction is given below.

The construction, if $B \neq \mathbf{C}$, depends on the theory of Hilbert C^* -modules; see Lance's book [6] for a good general reference. However, if $B = \mathbf{C}$ then all the Hilbert C^* -modules considered below become simply Hilbert spaces.

1.1. Let $E_\iota = L^2(A_\iota, \Phi_\iota)$ be the (right) Hilbert B -module obtained from A by separation and completion with respect to the norm $\|a\| = \|\langle a, a \rangle_{E_\iota}\|^{1/2}$, where $\langle \cdot, \cdot \rangle_{E_\iota}$ is the B -valued inner product, $\langle a_1, a_2 \rangle_{E_\iota} = \phi_\iota(a_1^* a_2)$; note that this inner product is conjugate linear in the first variable, as will be all the inner products in this paper. We denote the map $A_\iota \rightarrow E_\iota$ arising from the definition by $a \mapsto \widehat{a}$. Let $\pi_\iota : A_\iota \rightarrow \mathcal{L}(E_\iota)$ denote the $*$ -representation defined by $\pi_\iota(a)\widehat{b} = \widehat{ab}$, where as usual, for a Hilbert B -module E , $\mathcal{L}(E)$ denotes the C^* -algebra of all adjointable bounded B -module operators on E ; we will also let $\mathcal{K}(E)$ denote the C^* -subalgebra (in fact, the ideal) of $\mathcal{L}(E)$ generated by the collection of all operators of the form $\theta_{x,y}$ for $x, y \in E$, given by $\theta_{x,y}(e) = x\langle y, e \rangle_E$. Note that the condition (4) is equivalent to faithfulness of π_ι . Consider the specified element $\xi_\iota = \widehat{1_{A_\iota}} \in E_\iota$. We will call $(\pi_\iota, E_\iota, \xi_\iota)$ the GNS representation of (A_ι, ϕ_ι) and write $(\pi_\iota, E_\iota, \xi_\iota) = \text{GNS}(A_\iota, \phi_\iota)$, though this is technically a misnomer unless $B = \mathbf{C}$.

1.2. Voiculescu's construction of A proceeds via the construction of a Hilbert B -module E , which can later be seen to be $L^2(A, \phi)$. Note that the subspace $\xi_\iota B$ is a complemented submodule of E_ι that is invariant under the left action $\pi_\iota|_B$ of B ; indeed, $\theta_{\xi_\iota, \xi_\iota}$ is the projection onto $\xi_\iota B$. We will denote the complementing submodule by $E_\iota^\circ = P_\iota^\circ E_\iota$, where we define $P_\iota^\circ = 1 - \theta_{\xi_\iota, \xi_\iota} \in \mathcal{L}(E_\iota)$. Let

$$E = \xi B \oplus \bigoplus_{\substack{n \in \mathbf{N} \\ \iota_1, \dots, \iota_n \in I \\ \iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n}} E_{\iota_1}^\circ \otimes_B E_{\iota_2}^\circ \otimes_B \cdots \otimes_B E_{\iota_n}^\circ;$$

here ξB denotes the C^* -algebra B considered as a Hilbert B -module and with specified element $\xi = 1_B$, the tensor products are internal tensor products arising in relation to the $*$ -homomorphisms $P_\iota^\circ \pi_\iota|_B(\cdot) P_\iota^\circ$ from B to $\mathcal{L}(E_\iota^\circ)$, and \mathbf{N} for us always means the positive integers excluding 0. All of the tensor products denoted \otimes_B in this paper will be internal tensor products defined with respect to some $*$ -homomorphisms from B into $\mathcal{L}(F)$ for the various Hilbert B -modules F , these $*$ -homomorphisms arising canonically from $\pi|_B$ and the several constructions employed on Hilbert B -modules. The Hilbert B -module E constructed

above is call the *free product* of the E_ι with respect to specified vectors ξ_ι , and will be denoted by $(E, \xi) = \ast_{\iota \in I} (E_\iota, \xi_\iota)$.

For $\iota \in I$ let

$$E(\iota) = \eta_\iota B \oplus \bigoplus_{\substack{n \in \mathbf{N} \\ \iota_1, \dots, \iota_n \in I \\ \iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n \\ \iota_1 \neq \iota}} E_{\iota_1}^\circ \otimes_B E_{\iota_2}^\circ \otimes_B \cdots \otimes_B E_{\iota_n}^\circ,$$

where $\eta_\iota B$ is a copy of the Hilbert B -module B with $\eta_\iota = 1_B$, and let

$$V_\iota : E_\iota \otimes_B E(\iota) \rightarrow E \quad (5)$$

be the unitary operator defined as follows. In order to distinguish the tensor product in (5) from those appearing in elements of E and $E(\iota)$, we will use the symbol $\overset{\circ}{\otimes}$ for it; then V_ι is given by

$$\begin{aligned} V_\iota : \xi_\iota \overset{\circ}{\otimes} \eta_\iota &\mapsto \xi \\ \zeta \overset{\circ}{\otimes} \eta_\iota &\mapsto \zeta \\ \xi_\iota \overset{\circ}{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n) &\mapsto \zeta_1 \otimes \cdots \otimes \zeta_n \\ \zeta \overset{\circ}{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n) &\mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n, \end{aligned}$$

whenever $\zeta \in E_\iota^\circ$ and $\zeta_j \in E_{\iota_j}^\circ$ with $\iota \neq \iota_1, \iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n$. Let $\lambda_\iota : A_\iota \rightarrow \mathcal{L}(E)$ be the \ast -homomorphism given by

$$\lambda_\iota(a) = V_\iota(\pi_\iota(a) \otimes 1)V_\iota^*.$$

Then A is defined to be the C^* -algebra generated by $\bigcup_{\iota \in I} \lambda_\iota(A_\iota)$, and $\phi : A \rightarrow B$ is the conditional expectation $\phi(\cdot) = \langle \xi, \cdot \xi \rangle_E$. It is important to note that for $b \in B$, the operator $\lambda_\iota(b)$ on E does not depend on ι .

Let $A_\iota^\circ = A_\iota \cap \ker \phi_\iota$. Note that if $a \in A_\iota^\circ$ and if $\zeta_j \in E_{\iota_j}^\circ$ for $\iota_1, \dots, \iota_n \in I$, $n \geq 2$, and $\iota_j \neq \iota_{j+1}$, then

$$\lambda_\iota(a)(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} \widehat{a} \otimes \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n & \text{if } \iota \neq \iota_1 \\ (a\zeta_1 - \xi_{\iota_1} \langle \xi_{\iota_1}, a\zeta_1 \rangle) \otimes \zeta_2 \otimes \cdots \otimes \zeta_n \\ \quad + \pi_{\iota_2}(\langle \xi_{\iota_1}, a\zeta_1 \rangle) \zeta_2 \otimes \cdots \otimes \zeta_n & \text{if } \iota = \iota_1. \end{cases} \quad (6)$$

1.3. Throughout this paper we will usually omit to write the representations λ_ι and π_ι , and we will simply think of the A_ι as C^* -subalgebras of $\mathcal{L}(E_\iota)$ when it suits us and of $\mathcal{L}(E)$ when it suits us, and of B acting on the left and the right of just about everything.

1.4. The free product C^* -algebra A is the closed linear span of B together with the set of all *reduced words* of the form $w = a_1 a_2 \cdots a_q$ where $q \in \mathbf{N}$, $a_j \in A_{\iota_j}^\circ$, $\iota_1, \dots, \iota_q \in I$ and $\iota_1 \neq \iota_2, \dots, \iota_{q-1} \neq \iota_q$. The *length* of this word w is q . Using (6) one can see that if $w = a_1 \cdots a_q$ is a reduced word of length q and if $\zeta_1 \otimes \cdots \otimes \zeta_n \in E_{\iota_1}^\circ \otimes_B \cdots \otimes_B E_{\iota_n}^\circ \subseteq E$ for $n > q$, then when acting on $\zeta_1 \otimes \cdots \otimes \zeta_n$, w only sees the first bit, $\zeta_1 \otimes \cdots \otimes \zeta_q$, and $w(\zeta_1 \otimes \cdots \otimes \zeta_n)$ is a linear combination of simple tensors having lengths between $n - q$ and $n + q$, and each with the same tail $\cdots \otimes \zeta_{q+1} \otimes \cdots \otimes \zeta_n$, (possibly multiplied on the left by an element of B). The tail simply hangs on for the ride, so to speak.

§2. PRELIMINARY LEMMAS AND A CONSTRUCTION

In this section are assembled a few preliminary results and a construction involving Hilbert C^* -modules. We begin with an easy generalization of the proof of Wassermann's result [11] that nuclear embeddability implies exactness.

Lemma 2.1. *Let A' be a C^* -algebra and let A be a C^* -subalgebra of A' . Suppose that for every finite subset $\omega \subseteq A$ and every $\epsilon > 0$ there is an exact C^* -algebra D and there are completely positive contractions, $\Phi : A \rightarrow D$ and $\Psi : D \rightarrow A'$, such that $\|x - \Psi \circ \Phi(x)\| < \epsilon$ for every $x \in \omega$. Then A is exact.*

Proof. Using the hypotheses and taking the directed set of all finite subsets of A , one constructs a net $(\Phi_\lambda, D_\lambda, \Psi_\lambda)$ of exact C^* -algebras D_λ and completely positive maps $\Phi_\lambda : A \rightarrow D_\lambda$ and $\Psi_\lambda : D_\lambda \rightarrow A'$ such that

$$\forall x \in A \quad \lim_\lambda \|\Psi_\lambda \circ \Phi_\lambda(x) - x\| = 0.$$

Let

$$0 \rightarrow I \rightarrow B \xrightarrow{\pi} C \rightarrow 0$$

be an exact sequence of C^* -algebras and $*$ -homomorphisms. Let $x \in (A \otimes B) \cap \ker(\text{id}_A \otimes \pi)$.

We will show that $x \in A \otimes I$, which will prove the lemma. For every λ we have

$$0 = (\Phi_\lambda \otimes \text{id}_C) \circ (\text{id}_A \otimes \pi)(x) = (\text{id}_{D_\lambda} \otimes \pi) \circ (\Phi_\lambda \otimes \text{id}_B)(x),$$

so $(\Phi_\lambda \otimes \text{id}_B)(x) \in (D_\lambda \otimes B) \cap \ker(\text{id}_{D_\lambda} \otimes \pi)$. Since D_λ is exact, this implies that $(\Phi_\lambda \otimes \text{id}_B)(x) \in D_\lambda \otimes I$. Hence $((\Psi_\lambda \circ \Phi_\lambda) \otimes \text{id}_B)(x) \in A' \otimes I$. We then have that

$$x = \lim_{\lambda} ((\Psi_\lambda \circ \Phi_\lambda) \otimes \text{id}_B)(x) \in A' \otimes I.$$

But it is easily seen, using an approximate identity for I , that $(A \otimes B) \cap (A' \otimes I) = A \otimes I$. Therefore we find that $x \in A \otimes I$, as required. \square

The next result is well known to experts; proofs can be found at [2, Prop. 2] and [4, 7.1], and another proof is possible using exact operator spaces [7]. I would like to thank Etienne Blanchard for first bringing it to my attention and showing me a proof. Here and everywhere in this paper, ideals of C^* -algebras are closed, two-sided ideals.

Lemma 2.2. *Let A be a C^* -algebra, let J be an ideal of A and let $q : A \rightarrow A/J$ be the quotient map. Suppose that the short exact sequence*

$$0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$$

has a completely positive contractive splitting, i.e. a completely positive contraction, $s : A/J \rightarrow A$, such that $q \circ s = \text{id}_{A/J}$. If A/J and J are exact C^ -algebras, then A is an exact C^* -algebra.*

Lemma 2.3. *Let A be a unital C^* -algebra having a unital C^* -subalgebra B and a conditional expectation $\phi : A \rightarrow B$. Let $C(\mathbf{T})$ be the C^* -algebra of all continuous functions on the circle and let τ be the tracial state on $C(\mathbf{T})$ given by integration with respect to Haar measure. Consider the conditional expectation*

$$\phi \otimes \tau : A \otimes C(\mathbf{T}) \rightarrow B \otimes 1 \cong B$$

and let (π, F, ξ) be the GNS representation of $(A \otimes C(\mathbf{T}), \phi \otimes \tau)$ as described in 1.1. Then

$$\pi(A \otimes C(\mathbf{T})) \cap \mathcal{K}(F) = \{0\}.$$

Proof. Let

$$(\sigma, E, \eta) = \text{GNS}(A, \phi)$$

$$(\rho, \mathcal{H}, \eta') = \text{GNS}(C(\mathbf{T}), \tau);$$

then $\mathcal{H} = L^2(\mathbf{T})$ and $\rho(f)$, for $f \in C(\mathbf{T})$, is multiplication by f . The Hilbert $A \otimes C(\mathbf{T})$ -module F is canonically isomorphic to the external tensor product $E \otimes \mathcal{H}$, π is thereby identified with

$\sigma \otimes \rho$ and $\mathcal{K}(F)$ is identified with $\mathcal{K}(E) \otimes \mathcal{K}(\mathcal{H})$. But $\pi(A \otimes C(\mathbf{T}))$ commutes with $1 \otimes \rho(C(\mathbf{T}))$, and an elementary argument shows that

$$(\mathcal{L}(E) \otimes \mathcal{K}(\mathcal{H})) \cap (1 \otimes \rho(C(\mathbf{T})))' = \{0\}.$$

□

Lemma 2.4. *Let B be a C^* -algebra and let I be a countable set. For every $\iota \in I$ let A_ι be a C^* -algebra containing B as a C^* -subalgebra and having a conditional expectation $\phi_\iota : A_\iota \rightarrow B$. Let Y be a countable subset of B and for every $\iota \in I$ let X_ι be a countable subset of A_ι . Then there are separable C^* -subalgebras $D \subseteq B$ and $C_\iota \subseteq A_\iota$ ($\iota \in I$), such that $Y \subseteq D$, $X_\iota \subseteq C_\iota$, $D \subseteq C_\iota$ and $\phi_\iota(C_\iota) = D$ for every $\iota \in I$.*

Proof. Let $D_1 = C^*(Y) \subseteq B$ and $C_{\iota,1} = C^*(X_\iota) \subseteq A_\iota$, and for $n \in \mathbf{N}$ define recursively

$$C_{\iota,n+1} = C^*(C_{\iota,n} \cup D_n) \quad \text{and} \quad D_{n+1} = C^*\left(D_n \cup \bigcup_{\iota \in I} \phi_\iota(C_{\iota,n+1})\right).$$

Finally, let

$$C_\iota = \overline{\bigcup_{n \geq 1} C_{\iota,n}} \quad \text{and} \quad D = \overline{\bigcup_{n \geq 1} D_n}.$$

□

Remark 2.5. If in Lemma 2.4 each A_ι is unital, having B as a unital subalgebra and if the GNS representation of each ϕ_ι is faithful, then consider the reduced amalgamated free product

$$(A, \phi) = \bigstar_{\iota \in I} (A_\iota, \phi_\iota). \tag{9}$$

For D and C_ι as constructed in the lemma, take the conditional expectations $\psi_\iota \stackrel{\text{def}}{=} \phi_\iota|_{C_\iota} : C_\iota \rightarrow D$ and the reduced amalgamated free product

$$(C, \phi) = \bigstar_{\iota \in I} (C_\iota, \psi_\iota). \tag{10}$$

Then by the main result of [1], the embeddings $C_\iota \hookrightarrow A_\iota$ extend to an embedding $C \hookrightarrow A$. From this it is easily seen that every reduced amalgamated free product C^* -algebra A as in (9) is the inductive limit of reduced amalgamated free products of separable subalgebras of the A_ι as in (10).

Lemma 2.6. *Let B be an exact C^* -algebra and let E be a countably generated Hilbert B -module. Then $\mathcal{K}(E)$ is an exact C^* -algebra.*

Proof. By Kasparov's stabilization lemma, E is a complemented C^* -submodule of $\mathcal{H}_B \stackrel{\text{def}}{=} \mathcal{H} \otimes B$, the external tensor product of a separable infinite dimensional Hilbert space \mathcal{H} and the Hilbert B -module B . Thus $\mathcal{K}(E)$ is a C^* -subalgebra of $\mathcal{K}(\mathcal{H}_B) \cong \mathcal{K}(\mathcal{H}) \otimes B$, which is exact. \square

For the rest of this section we concentrate on the following construction.

Definition 2.7. Let B_1 and B_2 be C^* -algebras and let E_i be a Hilbert B_i -module, ($i = 1, 2$); let $\pi : B_1 \rightarrow \mathcal{L}(E_2)$ be a $*$ -homomorphism and let $E_1 \otimes_\pi E_2$ be the associated internal tensor product. If $A \subseteq \mathcal{L}(E_2)$ is a C^* -subalgebra such that $\pi(B_1)A \subseteq A$, then let

$$\mathcal{K}(E_1) \bowtie_\pi A \tag{11}$$

be the C^* -subalgebra of $\mathcal{L}(E_1 \otimes_\pi E_2)$ generated by

$$\{\theta_e a \theta_{e'}^* \mid e, e' \in E_1, a \in A\}, \tag{12}$$

where $\theta_e \in \mathcal{L}(E_2, E_1 \otimes_\pi E_2)$ is the operator defined by $\theta_e(f) = e \otimes f$, (see [6, 4.6]).

By identifying $E_1 \otimes_\pi E_2$ with $E_1 \otimes_\pi A \otimes_A E_2$, where we regard π as also a $*$ -homomorphism from B_1 into $\mathcal{L}(A)$ for the Hilbert A -module A , one sees that $\mathcal{K}(E_1) \bowtie_\pi A$ is canonically isomorphic to $\mathcal{K}(E_1 \otimes_\pi A) \otimes 1_{E_2}$. However, we persist with the notation (11) because it seems more convenient for keeping track of the tensor product structure of the Hilbert modules. It may be helpful to realize that if $B_1 = \mathbf{C}$ and π is unital then $\mathcal{K}(E_1) \bowtie_\pi A$ is simply $\mathcal{K}(E_1) \otimes A$.

Some easy facts about this construction are collected in the lemma below.

Lemma 2.8.

- (i) $\mathcal{K}(E_1) \bowtie_\pi A$ is the closed linear span of the set (12).
- (ii) $\mathcal{K}(E_1) \bowtie_\pi \mathcal{K}(E_2) = \mathcal{K}(E_1 \otimes_\pi E_2)$.
- (iii) Let \mathcal{H}_{B_1} be the Hilbert B_1 -module that is the external tensor product $\mathcal{H} \otimes B_1$, where \mathcal{H} is a separable, infinite dimensional Hilbert space. Then $\mathcal{K}(\mathcal{H}_{B_1}) \bowtie_\pi A$ is canonically isomorphic to $\mathcal{K}(\mathcal{H}) \otimes \pi(B_1)A\pi(B_1)$.
- (iv) Suppose E_1 is countably generated. Then by the Kasparov stabilization theorem, E_1 is isomorphic to a complemented submodule of \mathcal{H}_{B_1} . Let P be the projection from \mathcal{H}_{B_1} onto E_1 . The inclusion $E_1 \hookrightarrow \mathcal{H}_{B_1}$ provides an inclusion $\mathcal{K}(E_1) \bowtie_\pi A \hookrightarrow \mathcal{K}(\mathcal{H}_{B_1}) \bowtie_\pi A$.

A and the map $\Phi(x) = (P \otimes 1)x(P \otimes 1)$ is a conditional expectation from $\mathcal{K}(\mathcal{H}_{B_1}) \rtimes_{\pi} A$ onto $\mathcal{K}(E_1) \rtimes_{\pi} A$.

Proof. (i) follows from $\theta_e^*, \theta_e = \pi(\langle e', e \rangle)$.

(ii) follows from $\theta_e \theta_{f,f'} \theta_{e'}^* = \theta_{e \otimes f, e' \otimes f'}$.

In (iii), the isomorphism is $\rho : \mathcal{K}(\mathcal{H}_{B_1}) \rtimes_{\pi} A \rightarrow \mathcal{K}(\mathcal{H}) \otimes \pi(B_1)A\pi(B_1)$ given by, for $v, v' \in \mathcal{H}$, $b, b' \in B$ and $a \in A$,

$$\rho(\theta_{v \otimes b} a \theta_{v' \otimes b'}) = \theta_{v, v'} \otimes \pi(b) a \pi(b').$$

□

Perhaps slightly less obvious facts about this construction are below.

Lemma 2.9. *Let J be an ideal of A and suppose that E_1 is countably generated; note that $\pi_1(B_1)J \subseteq J$.*

(i) *Then $\mathcal{K}(E_1) \rtimes_{\pi} J$ is an ideal of $\mathcal{K}(E_1) \rtimes_{\pi} A$ and the quotient C^* -algebra*

$$\frac{\mathcal{K}(E_1) \rtimes_{\pi} A}{\mathcal{K}(E_1) \rtimes_{\pi} J}$$

is isomorphic to a C^ -subalgebra of $\mathcal{K} \otimes (A/J)$, where \mathcal{K} is the algebra of all compact operators on a separable infinite dimensional Hilbert space.*

(ii) *If the short exact sequence*

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0 \tag{13}$$

has a completely positive contractive splitting $s : (A/J) \rightarrow A$ then the short exact sequence

$$0 \rightarrow \mathcal{K}(E_1) \rtimes_{\pi} J \rightarrow \mathcal{K}(E_1) \rtimes_{\pi} A \rightarrow \frac{\mathcal{K}(E_1) \rtimes_{\pi} A}{\mathcal{K}(E_1) \rtimes_{\pi} J} \rightarrow 0$$

has a completely positive contractive splitting

$$\tilde{s} : \frac{\mathcal{K}(E_1) \rtimes_{\pi} A}{\mathcal{K}(E_1) \rtimes_{\pi} J} \rightarrow \mathcal{K}(E_1) \rtimes_{\pi} A.$$

Proof. Let us write $D = \mathcal{K}(E_1) \rtimes_\pi A$ and $I = \mathcal{K}(E_1) \rtimes_\pi J$. It is clear that I is an ideal of D . Let $\rho : \mathcal{K}(\mathcal{H}_{B_1}) \rtimes_\pi A \rightarrow \mathcal{K} \otimes A$ be the isomorphism indicated in Lemma 2.8(iii) and let $\Phi : \mathcal{K}(\mathcal{H}_{B_1}) \rtimes_\pi A \rightarrow \mathcal{K}(E_1) \rtimes_\pi A$ be the conditional expectation in 2.8(iv). One easily verifies that $\rho(\mathcal{K}(\mathcal{H}_{B_1}) \rtimes_\pi J) = \mathcal{K} \otimes J \subseteq \mathcal{K} \otimes A$ and $\Phi(\mathcal{K}(\mathcal{H}_{B_1}) \rtimes_\pi J) = I$. Therefore we see that $\rho(D) \cap (\mathcal{K} \otimes J) = \rho(I)$, and hence that D/I is isomorphic to the image, call it C , of $\rho(D)$ in the quotient $(\mathcal{K} \otimes A)/(\mathcal{K} \otimes J) \cong \mathcal{K} \otimes (A/J)$. This proves (i).

If $s : A/J \rightarrow A$ is a completely positive contractive lifting of the short exact sequence (13), then $\text{id}_\mathcal{K} \otimes s : \mathcal{K} \otimes (A/J) \rightarrow \mathcal{K} \otimes A$ is a completely positive contraction. Let $q_D : \rho(D) \rightarrow C$ be the quotient map; we seek a completely positive splitting for the short exact sequence

$$0 \rightarrow I \rightarrow D \xrightarrow{q_D \circ \rho} C \rightarrow 0. \quad (14)$$

Let $x \in C \subseteq \mathcal{K} \otimes (A/J)$ and let $y = (\text{id}_\mathcal{K} \otimes s)(x)$. Then $y \in \rho(D) + (\mathcal{K} \otimes J)$; hence $\rho \circ \Phi \circ \rho^{-1}(y) - y \in \mathcal{K} \otimes J$. But $\rho \circ \Phi \circ \rho^{-1}(y) \in \rho(D)$ and $q_D \circ \rho \circ \Phi \circ \rho^{-1}(y) = x$. Hence

$$\tilde{s} = \Phi \circ \rho^{-1} \circ (\text{id}_\mathcal{K} \otimes s)|_C : C \rightarrow D$$

is the desired completely positive contractive splitting of (14). □

Remark 2.10. It is natural to ask whether the splitting \tilde{s} constructed above satisfies

$$\tilde{s}(\theta_e a \theta_{e'}^* + \mathcal{K}(E_1 \rtimes_\pi J)) = \theta_e s(a + J) \theta_{e'} \quad (15)$$

for every $a \in A$ and $e, e' \in E_1$. I don't know the answer to this question in general, but if s is assumed to satisfy the additional condition $s(\pi(b)a + J) = \pi(b)s(a + J)$ then it is straightforward to show that (15) holds.

Lemma 2.11. *We have*

$$(\mathcal{K}(E_1) \rtimes_\pi \mathcal{L}(E_2)) \bigcap (\mathcal{L}(E_1) \otimes 1_{E_2}) = \mathcal{K}(E_1) \otimes 1_{E_2}. \quad (16)$$

Proof. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $\mathcal{K}(E_1)$, where each u_λ is of the form $\sum_{i=1}^n \theta_{e_i, e'_i}$, ($n \in \mathbf{N}$, $e_i, e'_i \in E_1$). Note that $\theta_{e_i} 1_{E_2} \theta_{e'_i}^* = \theta_{e_i, e'_i} \otimes 1_{E_2}$, and $(u_\lambda \otimes 1_{E_2})_{\lambda \in \Lambda}$ is an approximate identity for $\mathcal{K}(E_1) \rtimes_\pi \mathcal{L}(E_2)$. If y belongs to the left-hand-side of (16) then $y = x \otimes 1_{E_2}$ for some $x \in \mathcal{L}(E_1)$. But also $y = \lim_\lambda (u_\lambda x u_\lambda \otimes 1_{E_2}) \in \mathcal{K}(E_1) \otimes 1_{E_2}$. □

§3. EXACTNESS OF FREE PRODUCT C*-ALGEBRAS

This section contains the proof of the main theorem, that every reduced amalgamated free product of exact C*-algebras is exact. Let

$$(A, \phi) = \ast_{\iota \in I} (A_\iota, \phi_\iota)$$

be a reduced amalgamated free product of C*-algebras and let E be the Hilbert B -module as described in §1. Consider the submodules of E ,

$$E_{(\rightarrow k)} = \xi B \oplus \bigoplus_{\substack{n \in \{1, 2, \dots, k\} \\ \iota_1, \dots, \iota_n \in I \\ \iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n}} E_{\iota_1}^\circ \otimes_B E_{\iota_2}^\circ \otimes_B \cdots \otimes_B E_{\iota_n}^\circ, \quad (k \in \mathbf{N})$$

and $E_{(\rightarrow 0)} = \xi B$. Let $P_{(\rightarrow k)}$ denote the projection from E onto $E_{(\rightarrow k)}$. Consider the completely positive unital maps

$$\Phi_k : \mathcal{L}(E) \rightarrow \mathcal{L}(E_{(\rightarrow k)})$$

obtained by compressing: $\Phi_k(x) = P_{(\rightarrow k)} x|_{E_{(\rightarrow k)}}$. In the following lemma we will find completely positive unital maps going the other way which are approximately left inverses on elements of $A \subseteq \mathcal{L}(E)$.

Lemma 3.1. *There are completely positive unital maps, $\Psi_k : \mathcal{L}(E_{(\rightarrow k)}) \rightarrow \mathcal{L}(E)$ such that $\lim_{k \rightarrow \infty} \|a - \Psi_k \circ \Phi_k(a)\| = 0$ for every $a \in A$.*

Proof. We will first define, for every two positive integers p and k with $p < k$, an isometry

$$V_{p,k} : E \rightarrow E_{(\rightarrow k)} \otimes_B E \tag{17}$$

belonging to $\mathcal{L}(E, E_{(\rightarrow k)} \otimes_B E)$. Let us use the symbol $\ddot{\otimes}$ for the tensor product in (17), in order to distinguish it from tensor products appearing in elements of $E_{(\rightarrow k)}$ or E . We let $V_{p,k}(\xi) = \xi \ddot{\otimes} \xi$ and

$$V_{p,k}(\xi_1 \otimes \cdots \otimes \xi_n) = \begin{cases} (\xi_1 \otimes \cdots \otimes \xi_n) \ddot{\otimes} \xi & \text{if } 1 \leq n \leq p \\ \left(\sum_{j=0}^{n-p-1} \frac{1}{\sqrt{k-p}} (\xi_1 \otimes \cdots \otimes \xi_{p+j}) \ddot{\otimes} (\xi_{p+j+1} \otimes \cdots \otimes \xi_n) \right. \\ \quad \left. + \frac{\sqrt{k-n}}{\sqrt{k-p}} (\xi_1 \otimes \cdots \otimes \xi_n) \ddot{\otimes} \xi \right) & \text{if } p < n \leq k \\ \sum_{j=0}^{k-p-1} \frac{1}{\sqrt{k-p}} (\xi_1 \otimes \cdots \otimes \xi_{p+j}) \ddot{\otimes} (\xi_{p+j+1} \otimes \cdots \otimes \xi_n) & \text{if } k < n, \end{cases}$$

whenever $n \in \mathbf{N}$ and $\zeta_j \in E_{\iota_j}^\circ$, for some $\iota_1, \dots, \iota_n \in I$ with $\iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n$. It is not difficult to check that $V_{p,k} \in \mathcal{L}(E, E_{(\rightarrow k)} \otimes_B E)$ and is an isometry.

Let $\Theta_{p,k} : \mathcal{L}(E_{(\rightarrow k)}) \rightarrow \mathcal{L}(E)$ be the completely positive unital map defined by

$$\Theta_{p,k}(x) = V_{p,k}^*(x \otimes 1)V_{p,k}.$$

We claim that

$$\|a - \Theta_{p,k} \circ \Phi_k(a)\| \rightarrow 0 \quad (18)$$

for every $a \in A$, as $k \rightarrow \infty$ and $p \rightarrow \infty$ in such a way that $(k - p) \rightarrow \infty$. It is straightforward to show that $\Theta_{p,k} \circ \Phi_k(b) = b$ for every $b \in B \subseteq A$; hence, to show (18) for every $a \in A$, it will suffice to show it for every reduced word, $a = a_1 \cdots a_q$, (see 1.4).

Let us now consider some more submodules of E ; for $n \in \mathbf{N}$ let

$$E_{(n)} = \bigoplus_{\substack{\iota_1, \dots, \iota_n \in I \\ \iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n}} E_{\iota_1}^\circ \otimes_B E_{\iota_2}^\circ \otimes_B \cdots \otimes_B E_{\iota_n}^\circ,$$

and let $E_{(0)} = \xi B$; let $P_{(n)}$ be the projection from E onto $E_{(n)}$. We can think of operators, $x \in \mathcal{L}(E)$, as infinite matrices indexed by $\{0\} \cup \mathbf{N}$, where the (n, m) th entry is $P_{(n)}xP_{(m)}$. We will let $S_d(x)$ be the matrix consisting of only the d th diagonal:

$$S_d(x) = \sum_{n=\max(0, -d)}^{\infty} P_{(n+d)}xP_{(n)},$$

where the sum converges in the strict topology. Note that $\|S_d(x)\| \leq \|x\|$. Let $a = a_1 \cdots a_q$ be a reduced word of length q , $a_j \in A_{\iota_j}^\circ$; then $S_d(a)$ vanishes whenever $|d| > q$; thus $a = \sum_{d=-q}^q S_d(a)$. We will show that for every $d \in \{-q, -q+1, \dots, q\}$,

$$\|S_d(a) - \Theta_{p,k} \circ \Phi_k(S_d(a))\| \rightarrow 0$$

as k, p , and $k - p$ all tend to infinity, which will suffice to prove (18). Fix d and write $y = S_d(a)$ for convenience. We may and do assume that $p > 2q$ and $k - p > 2q$. Checking the several cases, one shows that $y - \Theta_{p,k} \circ \Phi_k(y) = yR$ where $R \in \mathcal{L}(E)$ is the operator that multiplies

ξ by 0 and every element in $E_{(n)}$ by a real number R_n , given by

$$R_n = \begin{cases} 0 & \text{if } n \leq p \text{ and } n + d \leq p \\ \left(1 - \sqrt{\frac{k-n-d}{k-p}}\right) & \text{if } n \leq p < n + d \leq k \\ \left(1 - \sqrt{\frac{k-n}{k-p}}\right) & \text{if } n + d \leq p < n \leq k \\ \left(\frac{k-n-\min(0,d)-(k-n)^{1/2}(k-n-d)^{1/2}}{k-p}\right) & \text{if } p < n \leq k \text{ and } p < n + d \leq k \\ \left(\frac{d}{k-p}\right) & \text{if } p < n \leq k < n + d \\ \left(\frac{-d-\sqrt{k-n-d}}{k-p}\right) & \text{if } p < n + d \leq k < n \\ \left(\frac{|d|}{k-p}\right) & \text{if } k < n \text{ and } k < n + d. \end{cases}$$

Let us write down this computation in the case where $p < n \leq k$ and $p < n + d \leq k$. Let $\iota_1, \dots, \iota_n \in I$ satisfy $\iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n$ and let $\zeta_j \in E_{\iota_j}^o$. We will investigate

$$(y - \Theta_{p,k} \circ \Phi_k(y))(\zeta_1 \otimes \dots \otimes \zeta_n) = (y - V_{p,k}^*(\Phi_k(y) \otimes 1)V_{p,k})(\zeta_1 \otimes \dots \otimes \zeta_n).$$

We have

$$\begin{aligned} V_{p,k}(\zeta_1 \otimes \dots \otimes \zeta_n) &= \left(\sum_{j=0}^{n-p-1} \frac{1}{\sqrt{k-p}} (\zeta_1 \otimes \dots \otimes \zeta_{p+j}) \ddot{\otimes} (\zeta_{p+j+1} \otimes \dots \otimes \zeta_n) \right) \\ &\quad + \frac{\sqrt{k-n}}{\sqrt{k-p}} (\zeta_1 \otimes \dots \otimes \zeta_n) \ddot{\otimes} \xi \end{aligned}$$

and for every $j \in \{0, 1, \dots, n-p\}$,

$$\Phi_k(y)(\zeta_1 \otimes \dots \otimes \zeta_{p+j}) = y(\zeta_1 \otimes \dots \otimes \zeta_{p+j}) \in E_{(p+j+d)}.$$

If $0 \leq j < -d$ then $p + j + d < p$, so

$$V_{p,k}^*(y(\zeta_1 \otimes \dots \otimes \zeta_{p+j}) \ddot{\otimes} (\zeta_{p+j+1} \otimes \dots \otimes \zeta_n)) = 0.$$

If $\max(-d, 0) \leq j < n - p$ then since $p > q$ and using 1.4 we have

$$\begin{aligned} V_{p,k}^*(y(\zeta_1 \otimes \dots \otimes \zeta_{p+j}) \ddot{\otimes} (\zeta_{p+j+1} \otimes \dots \otimes \zeta_n)) &= \\ &= \frac{1}{\sqrt{k-p}} (y(\zeta_1 \otimes \dots \otimes \zeta_{p+j}) \otimes (\zeta_{p+j+1} \otimes \dots \otimes \zeta_n)) \\ &= \frac{1}{\sqrt{k-p}} y(\zeta_1 \otimes \dots \otimes \zeta_n). \end{aligned}$$

Finally,

$$V_{p,k}^*(y(\zeta_1 \otimes \dots \otimes \zeta_n) \ddot{\otimes} \xi) = \frac{\sqrt{k-n-d}}{\sqrt{k-p}} y(\zeta_1 \otimes \dots \otimes \zeta_n).$$

Putting this together we get

$$(y - \Theta_{p,k} \circ \Phi_k(y))|_{E(n)} = \left(\frac{k - n - \min(0, d) - (k - n)^{1/2}(k - n - d)^{1/2}}{k - p} \right) y|_{E(n)},$$

as required. The other six cases are similar but a bit easier to check.

Since $|d| \leq q$ and q is fixed, we see that R_n tends to zero uniformly in n as p , k and $k - p$ all tend to infinity. We have shown (18). Letting $\Psi_k = \Theta_{[k/2],k}$ finishes the proof. \square

In the reduced amalgamated free product $(A, \phi) = \ast_{\iota \in I} (A_\iota, \phi_\iota)$ from the above lemma, if we assume that I is finite and each of the C^* -algebras A_ι is finite dimensional, then each of the submodules $E_{(\rightarrow k)}$ is finite dimensional; it follows that A is nuclearly embeddable, and hence is exact; this allows an alternative proof of Kirchberg's result [4, 7.2] that reduced amalgamated free products of finite dimensional C^* -algebras are exact. In order to prove our main result, we will use the notion analogous to nuclear embeddability as found in Lemma 2.1 and we will prove, for all k , exactness of a C^* -algebra containing $\Phi_k(A)$.

Theorem 3.2. *Suppose that B is a unital exact C^* -algebra, I is a set and for every $\iota \in I$ A_ι is a unital exact C^* -algebra containing B as a unital C^* -subalgebra and having a conditional expectation, ϕ_ι , from A_ι onto B , whose GNS representation is faithful. Let*

$$(A, \phi) = \ast_{\iota \in I} (A_\iota, \phi_\iota)$$

be the reduced amalgamated free product of C^ -algebras. Then A is exact.*

Proof. A is the inductive limit of C^* -algebras obtained by taking free products of finite subfamilies of $((A_\iota, \phi_\iota))_{\iota \in I}$; to see this, one can prove directly that the free product of a subfamily embeds in the free product of the larger family by examining the Hilbert C^* -modules on which these C^* -algebras act; this also follows from the more general result in [1]. Since exactness is preserved under taking inductive limits, we may without loss of generality assume that I is finite. Similarly, using Lemma 2.4 and [1] as described in Remark 2.5, we may without loss of generality assume that B and every A_ι is separable.

We shall use the same subspaces and completely positive maps as in the proof of Lemma 3.1, and the same notation. Recall also that P_ι° is the projection from E_ι onto E_ι° .

It may happen that, $A_\iota \cap \mathcal{K}(E_\iota) \neq \{0\}$; for technical reasons we would like to avoid this situation. Let

$$(D, \psi) = \ast_{\iota \in I} (A_\iota \otimes C(\mathbf{T}), \phi_\iota \otimes \tau),$$

where $C(\mathbf{T})$ the C^* -algebra of all continuous functions on the circle and where τ is the state on $C(\mathbf{T})$ given by integration with respect to Haar measure. It is fairly straightforward in this example (and similar ones involving tensor products) to show directly that A is isomorphic to a C^* -subalgebra of D ; moreover, this follows from the more general result found in [1]. Hence in order to show that A is exact it will suffice to show that D is exact. But each C^* -algebra $A_\iota \otimes C(\mathbf{T})$ is exact and acting on $L^2(A_\iota \otimes C(\mathbf{T}), \phi_\iota, \otimes \tau)$ contains no nonzero compact operators, by Lemma 2.3. Therefore, we may without loss of generality assume that $A_\iota \cap \mathcal{K}(E_\iota) = \{0\}$ for all $\iota \in I$.

For $\iota \in I$ let \widehat{A}_ι be the C^* -subalgebra of $\mathcal{L}(E_\iota^\circ)$ generated by $\{P_\iota^\circ a P_\iota^\circ \mid a \in A_\iota\} \cup \mathcal{K}(E_\iota^\circ)$. We will now show that there is an exact sequence

$$0 \rightarrow \mathcal{K}(E_\iota^\circ) \rightarrow \widehat{A}_\iota \rightarrow A_\iota \rightarrow 0 \quad (19)$$

with a completely positive unital splitting. Let

$$q_\iota^\circ : \mathcal{L}(E_\iota^\circ) \rightarrow \mathcal{L}(E_\iota^\circ) / \mathcal{K}(E_\iota^\circ)$$

be the quotient map. Since $\mathcal{K}(E_\iota^\circ) \subseteq \widehat{A}_\iota$, we have the short exact sequence

$$0 \rightarrow \mathcal{K}(E_\iota^\circ) \rightarrow \widehat{A}_\iota \xrightarrow{q_\iota^\circ} q_\iota^\circ(\widehat{A}_\iota) \rightarrow 0;$$

let us show that $q_\iota^\circ(\widehat{A}_\iota) \cong A_\iota$. Clearly $q_\iota^\circ(\widehat{A}_\iota)$ is generated by $\{q_\iota^\circ(P_\iota^\circ a P_\iota^\circ) \mid a \in A_\iota\}$. Since $P_\iota^\circ = 1 - \theta_{\xi_\iota, \xi_\iota}$,

$$(P_\iota^\circ a_1 P_\iota^\circ)(P_\iota^\circ a_2 P_\iota^\circ) - (P_\iota^\circ a_1 a_2 P_\iota^\circ)$$

is compact for all $a_1, a_2 \in A_\iota$. Therefore, the map

$$A_\iota \ni a \mapsto q_\iota^\circ(P_\iota^\circ a P_\iota^\circ) \quad (20)$$

is a $*$ -homomorphism onto $q_\iota^\circ(\widehat{A}_\iota)$. But if $a \in A_\iota$ and $P_\iota^\circ a P_\iota^\circ$ is compact then $a \in \mathcal{K}(E_\iota)$ and hence $a = 0$ by assumption. Therefore the map in (20) is an isomorphism from A_ι to $q_\iota^\circ(\widehat{A}_\iota)$. Now it is clear that the map

$$A_\iota \ni a \mapsto P_\iota^\circ a P_\iota^\circ \in \widehat{A}_\iota$$

is a completely positive unital splitting of the exact sequence (19).

Fix $k \in \mathbb{N}$ and let us investigate $\Phi_k(A)$. For $p \in \{1, 2, \dots, k\}$ and $\iota \in I$ consider the Hilbert B -modules

$$L_{p,\iota} = \begin{cases} \eta B & \text{if } p = k \\ \eta B \oplus \bigoplus_{\substack{\ell \in \{1, 2, \dots, k-p\} \\ \iota_1, \dots, \iota_\ell \in I \\ \iota_1 \neq \iota_2, \dots, \iota_{\ell-1} \neq \iota_\ell \\ \iota_\ell \neq \iota}} E_{\iota_1}^\circ \otimes_B \dots \otimes_B E_{\iota_\ell}^\circ & \text{if } p < k \end{cases}$$

$$R_{p,\iota} = \begin{cases} \eta B & \text{if } p = 1 \\ \bigoplus_{\substack{\iota_1, \dots, \iota_{p-1} \in I \\ \iota_1 \neq \iota_2, \dots, \iota_{p-2} \neq \iota_{p-1} \\ \iota_1 \neq \iota}} E_{\iota_1}^\circ \otimes_B \dots \otimes_B E_{\iota_{p-1}}^\circ & \text{if } p > 1; \end{cases}$$

As usual, ηB refers to a copy of the Hilbert B -module B where the identity element of B is named η . Let $V_{p,\iota} \in \mathcal{L}(L_{p,\iota} \otimes_B E_\iota^\circ \otimes_B R_{p,\iota}, E)$ be the isometry defined by erasing parenthesis and absorbing all occurrences of η ; namely, given $e \in E_\iota^\circ$, $\zeta_j \in E_{\iota_j}^\circ$ and $\zeta'_j \in E_{\iota'_j}^\circ$ for $\iota_1, \dots, \iota_\ell, \iota'_1, \dots, \iota'_{p-1} \in I$ with $\iota_j \neq \iota_{j+1}$, $\iota'_j \neq \iota'_{j+1}$, and using $\ddot{\otimes}$ for the tensor product symbols in $L_{p,\iota} \otimes_B E_\iota^\circ \otimes_B R_{p,\iota}$, we let

$$\begin{aligned} V_{p,\iota} : \eta \ddot{\otimes} e \ddot{\otimes} \eta &\mapsto e & (\text{if } p = 1) \\ \eta \ddot{\otimes} e \ddot{\otimes} (\zeta'_1 \otimes \dots \otimes \zeta'_{p-1}) &\mapsto e \otimes \zeta'_1 \otimes \dots \otimes \zeta'_{p-1} & (\text{if } p > 1) \\ (\zeta_1 \otimes \dots \otimes \zeta_\ell) \ddot{\otimes} e \ddot{\otimes} \eta &\mapsto \zeta_1 \otimes \dots \otimes \zeta_\ell \otimes e & (\text{if } p = 1) \\ (\zeta_1 \otimes \dots \otimes \zeta_\ell) \ddot{\otimes} e \ddot{\otimes} (\zeta'_1 \otimes \dots \otimes \zeta'_{p-1}) &\mapsto \zeta_1 \otimes \dots \otimes \zeta_\ell \otimes e \otimes \zeta'_1 \otimes \dots \otimes \zeta'_{p-1} & (\text{if } p > 1). \end{aligned}$$

Note that for fixed p , $(V_{p,\iota}(L_{p,\iota} \otimes_B E_\iota^\circ \otimes_B R_{p,\iota})V_{p,\iota}^*)_{\iota \in I}$ is a family of mutually orthogonal complemented subspaces of $E_{(\rightarrow k)}$.

Let $a = a_1 a_2 \dots a_q \in A$, where $a_j \in A^\circ_{\iota_j}$, $\iota_1 \neq \iota_2, \dots, \iota_{q-1} \neq \iota_q$. Let $\zeta_j \in E_{\iota'_j}^\circ$ ($1 \leq j \leq n$) with $\iota'_1 \neq \iota'_2, \dots, \iota'_{n-1} \neq \iota'_n$. If $\iota_q \neq \iota'_1$ then

$$a(\zeta_1 \otimes \dots \otimes \zeta_n) = \widehat{a_1} \otimes \dots \otimes \widehat{a_q} \otimes \zeta_1 \otimes \dots \otimes \zeta_n.$$

If $\iota_q = \iota'_1$ and $\iota_{q-1} \neq \iota'_2$ then

$$\begin{aligned} a(\zeta_1 \otimes \dots \otimes \zeta_n) &= \widehat{a_1} \otimes \dots \otimes \widehat{a_{q-1}} \otimes \langle \widehat{a_q^*}, \zeta_1 \rangle \zeta_2 \otimes \zeta_3 \otimes \dots \otimes \zeta_n \\ &\quad + \widehat{a_1} \otimes \dots \otimes \widehat{a_{q-1}} \otimes (P_{\iota_q}^\circ a_q \zeta_1) \otimes \zeta_2 \otimes \dots \otimes \zeta_n. \end{aligned}$$

Continuing in this way, and noting that, for example,

$$\langle \widehat{a_{q-1}^*}, \langle \widehat{a_q^*}, \zeta_1 \rangle \zeta_2 \rangle = \langle \widehat{a_q^*} \otimes \widehat{a_{q-1}^*}, \zeta_1 \otimes \zeta_2 \rangle,$$

we see that if $\iota_1 = \iota'_1, \iota_{q-1} = \iota'_2, \dots, \iota_{q-\ell+1} = \iota'_\ell$ but $\iota_{q-\ell} \neq \iota'_{\ell+1}$, some $\ell < \min(q, n)$, then

$$\begin{aligned} a(\zeta_1 \otimes \dots \otimes \zeta_n) &= \widehat{a_1} \otimes \dots \otimes \widehat{a_{q-\ell}} \otimes \langle \widehat{a_q^*} \otimes \dots \otimes \widehat{a_{q-\ell+1}^*}, \zeta_1 \otimes \dots \otimes \zeta_\ell \rangle \zeta_{\ell+1} \otimes \zeta_{\ell+2} \otimes \dots \otimes \zeta_n \\ &\quad + \sum_{j=2}^{\ell} \left(\widehat{a_1} \otimes \dots \otimes \widehat{a_{q-j}} \otimes \right. \\ &\quad \left. \otimes (P_{\iota_{q-j+1}}^o a_{q-j+1} \langle \widehat{a_q^*} \otimes \dots \otimes \widehat{a_{q-j+2}^*}, \zeta_1 \otimes \dots \otimes \zeta_{j-1} \rangle \zeta_j) \otimes \right. \\ &\quad \left. \otimes \zeta_{j+1} \otimes \dots \otimes \zeta_n \right) \\ &\quad + \widehat{a_1} \otimes \dots \otimes \widehat{a_{q-1}} \otimes (P_{\iota_q}^o a_q \zeta_1) \otimes \zeta_2 \otimes \dots \otimes \zeta_n. \end{aligned}$$

Making use of this and other related formulas and employing the convention that a sum $\sum_{j=n}^m x_j$ is zero whenever $m < n$, we find that for $a = a_1 a_2 \dots a_q$ as above,

$$\Phi_k(a) = \sum_{j=\max(0, q-k)}^{\min(q, k)} \theta_{\widehat{a_1} \otimes \dots \otimes \widehat{a_j}, \widehat{a_q^*} \otimes \dots \otimes \widehat{a_{j+1}^*}} \quad (21)$$

$$\begin{aligned} &+ \sum_{p=1}^{k-q} \left(\sum_{\iota \in I \setminus \{\iota_1\}} V_{p, \iota} \left((\theta_\eta 1_{E_\iota} \theta_{\widehat{a_q^*} \otimes \dots \otimes \widehat{a_1^*}}^*) \otimes 1_{R_{p, \iota}} \right) V_{p, \iota}^* \right. \\ &\quad \left. + \sum_{\iota \in I \setminus \{\iota_q\}} V_{p, \iota} \left((\theta_{\widehat{a_1} \otimes \dots \otimes \widehat{a_q}} 1_{E_\iota} \theta_\eta^*) \otimes 1_{R_{p, \iota}} \right) V_{p, \iota}^* \right) \\ &+ \sum_{p=1}^{\min(k-1, k-\lceil \frac{q+1}{2} \rceil)} \sum_{j=\max(1, p+q-k)}^{\min(q-1, k-p)} \sum_{\iota \in I \setminus \{\iota_j, \iota_{j+1}\}} V_{p, \iota} \left((\theta_{\widehat{a_1} \otimes \dots \otimes \widehat{a_j}} 1_{E_\iota} \theta_{\widehat{a_q^*} \otimes \dots \otimes \widehat{a_{j+1}^*}}^*) \otimes 1_{R_{p, \iota}} \right) V_{p, \iota}^* \\ &+ \sum_{p=1}^{k-\lceil \frac{q}{2} \rceil} \sum_{j=\max(1, p+q-k)}^{\min(q, k-p+1)} V_{p, \iota_j} \left((\theta_{\widehat{a_1} \otimes \dots \otimes \widehat{a_{j-1}}} P_{\iota_j}^o a_j P_{\iota_j}^o \theta_{\widehat{a_q^*} \otimes \dots \otimes \widehat{a_{j+1}^*}}^*) \otimes 1_{R_{p, \iota}} \right) V_{p, \iota_j}^*, \quad (22) \end{aligned}$$

where the symbols

- $\widehat{a_1} \otimes \dots \otimes \widehat{a_j}$ in line (21) should be interpreted to mean ξ if $j = 0$,
- $\widehat{a_q^*} \otimes \dots \otimes \widehat{a_{j+1}^*}$ in line (21) should be interpreted to mean ξ if $j = q$,
- $\widehat{a_1} \otimes \dots \otimes \widehat{a_{j-1}}$ in line (22) should be interpreted to mean η if $j = 1$,
- $\widehat{a_q^*} \otimes \dots \otimes \widehat{a_{j+1}^*}$ in line (22) should be interpreted to mean η if $j = q$.

Observe also that if $b \in B$ then

$$\Phi_k(b) = \theta_{b, \xi} + \sum_{p=1}^k \sum_{\iota \in I} V_{p, \iota} \left((\theta_\eta b \theta_\eta^*) \otimes 1_{R_{p, \iota}} \right) V_{p, \iota}^*. \quad (23)$$

The importance of these expressions is that they give $\Phi_k(a_1 \cdots a_q)$ and $\Phi_k(b)$ as sums of finitely many elements of

$$\mathcal{K}(E_{(\rightarrow k)}) \cup \bigcup_{\substack{p \in \{1, \dots, k\} \\ \iota \in I}} V_{p, \iota} ((\mathcal{K}(L_{p, \iota}) \rtimes \widehat{A}_\iota) \otimes 1_{R_{p, \iota}}) V_{p, \iota}^*. \quad (24)$$

The C^* -algebra generated by the set in (24) therefore contains $\Phi_k(A)$. We will find a sort of composition series decomposition of this C^* -algebra and thereby show that it is exact.

Given $p \in \{1, \dots, k\}$ let

$$D_p = \overline{\text{span}} \left(\bigcup_{\iota \in I} V_{p, \iota} ((\mathcal{K}(L_{p, \iota}) \rtimes \widehat{A}_\iota) \otimes 1_{R_{p, \iota}}) V_{p, \iota} \right).$$

Then D_p is a C^* -subalgebra of $\mathcal{L}(E_{(\rightarrow k)})$ that is isomorphic to

$$\bigoplus_{\iota \in I} \mathcal{K}(L_{p, \iota}) \rtimes \widehat{A}_\iota.$$

Moreover, it is not difficult to see that

$$D_{p_1} D_{p_2} \subseteq D_{\min(p_1, p_2)}. \quad (25)$$

Let $\mathcal{I}_0 = \mathcal{K}(E_{(\rightarrow k)})$ and for every $p \in \{1, \dots, k\}$ let $\mathcal{I}_p = \mathcal{I}_{p-1} + D_p$. Clearly (25) implies that \mathcal{I}_{p-1} is an ideal of \mathcal{I}_p . We will show by induction on $p \in \{0, 1, \dots, k\}$ that \mathcal{I}_p is a C^* -algebra and that for every $p \geq 1$ the short exact sequence

$$0 \rightarrow \mathcal{I}_{p-1} \rightarrow \mathcal{I}_p \rightarrow \mathcal{I}_p / \mathcal{I}_{p-1} \rightarrow 0 \quad (26)$$

has a completely positive contractive splitting and

$$\mathcal{I}_p / \mathcal{I}_{p-1} \cong \begin{cases} \text{a } C^*\text{-subalgebra of } \bigoplus_{\iota \in I} \mathcal{K} \otimes A_\iota & \text{if } p < k \\ \bigoplus_{\iota \in I} A_\iota & \text{if } p = k. \end{cases} \quad (27)$$

Elementary theory of rings implies that the quotient $*$ -algebra $\mathcal{I}_p / \mathcal{I}_{p-1}$ is isomorphic to $D_p / (D_p \cap \mathcal{I}_{p-1})$, and the isomorphism is seen to preserve the quotient norms. Since $D_p / (D_p \cap \mathcal{I}_{p-1})$ is a C^* -algebra, it follows that \mathcal{I}_p is a C^* -algebra.

We have the commuting diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{I}_{p-1} & \rightarrow & \mathcal{I}_p & \rightarrow & \mathcal{I}_p / \mathcal{I}_{p-1} \rightarrow 0 \\ & & \cup & & \cup & & \uparrow \\ 0 & \rightarrow & D_p \cap \mathcal{I}_{p-1} & \rightarrow & D_p & \rightarrow & D_p / (D_p \cap \mathcal{I}_{p-1}) \rightarrow 0. \end{array}$$

In order to find a completely positive contractive splitting for the sequence in the top row, it will suffice to find one for the sequence in the bottom row. Since the projection $V_{p,\iota} V_{p,\iota}^*$ commutes with D_p for each $\iota \in I$, it will suffice to find a completely positive contractive splitting for the exact sequence

$$0 \rightarrow V_{p,\iota}^*(D_p \cap \mathcal{I}_{p-1})V_{p,\iota} \rightarrow V_{p,\iota}^* D_p V_{p,\iota} \rightarrow (V_{p,\iota}^* D_p V_{p,\iota}) / (V_{p,\iota}^*(D_p \cap \mathcal{I}_{p-1})V_{p,\iota}) \rightarrow 0. \quad (28)$$

But

$$V_{p,\iota}^* D_p V_{p,\iota} = (\mathcal{K}(L_{p,\iota}) \rtimes \widehat{A}_\iota) \otimes 1_{R_{p,\iota}}$$

while

$$\mathcal{K}(L_{p,\iota} \otimes_B E_\iota^\circ) \otimes 1_{R_{p,\iota}} \subseteq V_{p,\iota}^*(D_p \cap \mathcal{I}_{p-1})V_{p,\iota} \subseteq \mathcal{K}(L_{p,\iota} \otimes_B E_\iota^\circ) \rtimes \mathcal{L}(R_{p,\iota}).$$

Clearly

$$\mathcal{K}(L_{p,\iota} \otimes_B E_\iota^\circ) \rtimes \mathcal{L}(R_{p,\iota}) = \mathcal{K}(L_{p,\iota}) \rtimes (\mathcal{K}(E_\iota^\circ) \rtimes \mathcal{L}(R_{p,\iota}))$$

and from Lemma 2.11 we have

$$(\mathcal{K}(E_\iota^\circ) \rtimes \mathcal{L}(R_{p,\iota})) \cap (\widehat{A}_\iota \otimes 1_{R_{p,\iota}}) = \mathcal{K}(E_\iota^\circ) \otimes 1_{R_{p,\iota}};$$

hence

$$V_{p,\iota}^*(D_p \cap \mathcal{I}_{p-1})V_{p,\iota} = (\mathcal{K}(L_{p,\iota}) \rtimes \mathcal{K}(E_\iota^\circ)) \otimes 1_{R_{p,\iota}}.$$

Therefore, using Lemma 2.9 we see that $D_p / (D_p \cap \mathcal{I}_{p-1})$ is isomorphic to a C^* -subalgebra of $\mathcal{K} \otimes A_\iota$, and, in light of the completely positive contractive splitting we found for the short exact sequence (19), the short exact sequence (28) has a completely positive contractive splitting. Putting these together, we obtain the isomorphisms (27) and a completely positive contractive splitting for (26).

We can now finish the proof of the theorem. By Lemma 2.6, the C^* -algebra \mathcal{I}_0 is exact. The isomorphisms (27) imply that the quotients $\mathcal{I}_{p-1} / \mathcal{I}_p$ are exact C^* -algebras. Using the completely positively contractively split exact sequences (26) and Lemma 2.2, we can prove by induction that \mathcal{I}_p is exact for all $p \in \{0, 1, \dots, k\}$. But since $\Phi_k(A)$ is in the C^* -algebra generated by (24), we see that $\Phi_k(A) \subseteq \mathcal{I}_k$. Therefore, we may use Lemma 2.1 together with the completely positive unital maps Φ_k and Ψ_k found in Lemma 3.1 to conclude that A is exact. □

Recall that a discrete group is exact if and only if its reduced group C^* -algebra is exact. The following corollary is a result of Theorem 3.2 and the fact, expressed in equation (3)

in the introduction, that certain reduced amalgamated free products of group C^* -algebras correspond to amalgamated free products of groups.

Corollary 3.3. *Suppose H is a group, I is a set and for every $\iota \in I$, G_ι is an exact group taken with the discrete topology and containing a copy of H as a subgroup. Let $G = (*_H)_{\iota \in I} G_\iota$ be the free product of groups with amalgamation over H . Then G is an exact group.*

REFERENCES

1. E. Blanchard, K.J. Dykema, *Embeddings of reduced free products of operator algebras*, preprint (1999).
2. P. de la Harpe, A.G. Robertson, A. Valette, *On exactness of group C^* -algebras*, Quart. J. Math. Oxford (2) **45** (1994), 499-513.
3. E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. **129** (1995), 35-63.
4. ———, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. reine angew. Math. **452** (1994), 39-77.
5. E. Kirchberg, S. Wassermann, *Exact groups and continuous bundles of C^* -algebras*, preprint.
6. E.C. Lance, *Hilbert C^* -modules, a Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Series, vol. 210, Cambridge University Press, 1995.
7. G. Pisier, *Exact operator spaces*, Recent Advances in Operator Algebras, Orléans 1992, Astérisque, vol. 232, Soc. Math. France, 1995, pp. 159-186.
8. D. Voiculescu, *Symmetries of some reduced free product C^* -algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, vol. 1132, Springer-Verlag, 1985, pp. 556-588.
9. D. Voiculescu, K.J. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series vol. 1, American Mathematical Society, 1992.
10. S. Wassermann, *On tensor products of certain group C^* -algebras*, J. Funct. Anal. **23** (1976), 239-254.
11. ———, *Tensor products of free-group C^* -algebras*, Bull. London Math. Soc. **22** (1990), 375-380.
12. ———, *Exact C^* -algebras and Related Topics*, Seoul National University Lecture Notes Series, vol. 19, 1994.

DEPT. OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION TX 77843-3368, USA
E-mail address: Ken.Dykema@math.tamu.edu, *Internet URL:* <http://www.math.tamu.edu/~Ken.Dykema/>